Optimal Accumulation of Private and Public Capital in an Endogenous Growth Model

Koichi Futagami*
Yuichi Morita
Akihisa Shibata

Abstract

Futagami, Morita and Shibata (1993) develop an endogenous growth model which includes public capital along with private capital. In their model, private agents do not care about the existence of the positive effect of public investment; therefore, the market equilibrium becomes suboptimal. This paper reexamines their analysis by assuming that the private agents take the positive effect into consideration. It is shown that there exists a unique steady growth equilibrium under certain mild conditions and that the steady growth equilibrium is saddle point stable. It is also shown that the long-run growth rate of this economy is higher than when externalities are ignored and that the long-run effects of changing the income tax rate are the same as in Futagami, Morita and Shibata.

1. Introduction

Barro (1990) investigates the effects of income tax on the long-run growth rate by using a simple endogenous growth model. In his model, government expenditure enters into the production function and is produc-

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tive. Barro considers that the public services which are flow variables as an input in private production function.

But, in the literature on public investment (for example, see Arrow and Kurz (1970)), it is commonly assumed that the *stock* of public capital is a productive input in the private production function. For example, highways, airports, and the electrical and gas facilities, and water systems would be productive and be modeled as public capital.[1)]

Futagami, Morita and Shibata (1993) take into account this argument and constitute a model in which the public capital is a productive input into the private production function.[2)] In contrast with Barro's model, since their model includes two state variables; private capital and public capital, it has transitional dynamics. They also reexamine Barro's optimal policy rule which states that the government should choose the tax rate that maximizes the national growth rate in order to maximize the welfare of the representative household. But the analysis in the paper is restricted to the private decision problem in which the representative household does not care about the productive effects of public capital. In other words, the public capital is associated with externalities in their model. Therefore, the privately determined economic growth becomes suboptimal. In this paper, we investigate the case in which the representative household takes the accumulation of public capital into account. We show that the same comparative static results as in Futagami, Morita and Shibata (1993) hold with respect to the steady-state equilibrium and that the privately determined growth rate is lower than the growth rate when the household takes the externalities into account.

The structure of this paper is as follows. Section 2 sets up the model. Section 3 characterizes the steady-growth equilibrium and proves the un-

1) Uzawa (1974) also investigates a model which includes social overhead capital and examines the pattern of resource allocation.

2) Mino (1990) and Lee (1992) investigate similar models like this paper. But the concern of their paper is the character of the steady-growth path. See also Barro and Sala-i-Martin (1992).

iqueness of the transitional path. Section 3 examines the effects of income taxation on the long-run growth rate and the steady-state equilibrium.

2. The Model

The representative infinitely-lived household maximizes the discounted sum of utility, as given by;

$$\int_0^\infty u(c)e^{-\rho t}dt \tag{1}$$

where c is consumption per person and ρ is the constant rate of time preference. In the following, we assume that population, which corresponds to the number of workers and consumers, is constant and that the instantaneous utility function takes the form of:

$$u(c)=\frac{c^{1-\sigma}-1}{1-\sigma},\ \ for\ \sigma>0,\ \ \sigma\neq 0$$

$$=\ln c,\ \ for\ \sigma=1,$$

where σ is positive constant and denotes the elasticity of marginal utility of consumption (an inverse of the elasticity of intertemporal substitution).

Following Barro, we formulate the production function as follows. The production function exhibits constant returns to scale with diminishing returns with respect to each factor:

$$q=f(k,\ g)=k\varphi(g/k),\ \ \varphi'>0,\ \ \varphi''<0, \tag{2}$$

where q is output, k is private capital and g is the quantity of public capital provided to the household-producer. q, k and g are measured in per capita terms. We assume that public capital is provided without user charges. Moreover, we assume that φ satisfies the Inada conditions:

$$\lim_{x\to 0}\varphi'(x)=\infty,\ \ \lim_{x\to\infty}\varphi'(x)=0, \tag{3}$$

where $x\equiv g/k$.

We assume that public expenditure consists of only public investment and is financed by flat-rate income tax:

$$\dot{g}=T=\tau q=\tau k\varphi(x), \tag{4}$$

where T is government revenue and τ is the tax rate. In order to maintain tractability, we follow Barro in making assumption that τ is time invariant, hence, the government does not choose the rate of income tax optimally.

The representative household maximizes (1) subject to (4) and

$$\dot{k}=(1-\tau)f(k,\ g)-c. \tag{5}$$

because he or she takes the accumulation of public capital into account.

In order to calculate the solution of this problem, we construct the Hamiltonian as follows:

$$H=\frac{c^{1-\sigma}-1}{(1-\sigma)}+\theta_1[(1-\tau)f(k,\ g)-c]+\theta_2\tau f(k,\ g).$$

The necessary conditions are

$$c^{-\sigma}=\theta_1, \tag{6}$$

$$\dot{\theta}_1=\rho\theta_1-\theta_1(1-\tau)f_k-\theta_2\tau f_k, \tag{7}$$

$$\dot{\theta}_2=\rho\theta_2-\theta_1(1-\tau)f_g-\theta_2\tau f_g. \tag{8}$$

with (4) and (5). Letting θ denote θ_2/θ_1, we can arrange the above necessary conditions as follows:

$$\frac{\dot{c}}{c}=\frac{1}{\sigma}[\{(1-\tau)+\theta\tau\}(\varphi-\varphi'x)-\rho], \tag{9}$$

$$\frac{\dot{k}}{k}=(1-\tau)\varphi-y, \tag{10}$$

$$\frac{\dot{g}}{g}=\frac{\tau\varphi}{x}, \tag{11}$$

$$\frac{\dot{\theta}}{\theta}=\frac{\dot{\theta}_2}{\theta_2}-\frac{\dot{\theta}_1}{\theta_1}=\left(\frac{1-\tau}{\theta}+\tau\right)[\theta(\varphi-\varphi'x)-\varphi'], \tag{12}$$

If the household does not take the accumulation of public capital into account, the term $\theta\tau$ in (9) does not appear. This term raises the marginal productivity of private capital in comparison with the market equilibrium growth path.

Summarizing these differential equations, we get the following dynamic system :

$$\frac{\dot{x}}{x}=\frac{\dot{g}}{g}-\frac{\dot{k}}{k}=\tau\frac{\varphi}{x}-(1-\tau)\varphi+y, \tag{13}$$

$$\frac{\dot{y}}{y}=\frac{\dot{c}}{c}-\frac{\dot{k}}{k}=\frac{1}{\sigma}[\{(1-\tau)+\theta\tau\}(\varphi-\varphi'x)-\rho]-(1-\tau)\varphi+y, \tag{14}$$

and (12). The stationary state equilibrium must satisfy the following conditions :

$$y^{*\circ}=\left[(1-\tau)-\frac{\tau}{x^{*\circ}}\right]\varphi(x^{*\circ}), \tag{15}$$

$$y^{*\circ}=(1-\tau)\varphi(x^{*\circ})-\frac{1}{\sigma}[\{(1-\tau)+\theta^{*\circ}\tau\}\{(\varphi(x^{*\circ})-\varphi'(x^{*\circ})x^{*\circ}\}-\rho], \tag{16}$$

$$\theta^{*\circ}=\frac{\varphi'(x^{*\circ})}{\varphi(x^{*\circ})-\varphi'(x^{*\circ})x^{*\circ}}, \tag{17}$$

where $x^{*\circ}$, $y^{*\circ}$ and $\theta^{*\circ}$ mean the steady state values of x, y and θ respectively. We call this steady-state equilibrium the planned steady-growth path.

Subtracting (16) from (15), we obtain the following :

$$\frac{1}{\sigma}[(1-\tau)(\varphi^*-\varphi^{*\prime}x^{*\circ})+\tau\varphi^{*\prime}]-\frac{\tau}{x^{*\circ}}\varphi^*-\frac{\rho}{\sigma}=0.$$

where φ^* means $\varphi(x^{*\circ})$. Let's denote the left hand side of this expression as $F(x)$. As can be easily verified, this function is a monotonically increasing function. Therefore, if a planned steady-growth path exists, it is unique. Moreover, if $\sigma\geq\eta$ then the followings hold :

$$\lim_{x\to 0}F(x)=-\infty,\quad \lim_{x\to\infty}F(x)=\infty.$$

On the other hand, if σ is smaller than η, F never takes a negative value, that is, there exists no steady-state equilibrium ; hence, we assume that $\sigma\geq\eta$ in the following analysis.

By using (17), second equation (16) can be rewritten as :

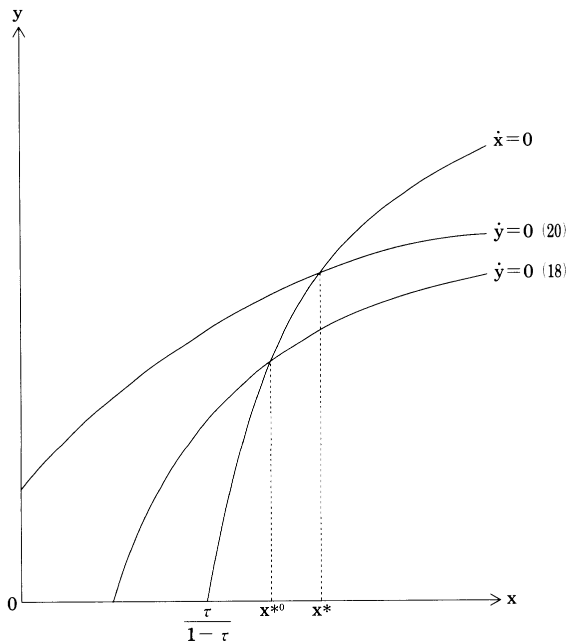
$$y^{*\circ}=\frac{1}{\sigma}\left[\left\{(1-\tau)(\sigma+\eta-1)-\eta\frac{\tau}{x^{*\circ}}\right\}\varphi(x^{*\circ})+\rho\right]. \tag{18}$$

If the household does not care about the accumulation of public capital, then the shadow price of public capital θ measured by that of private capital becomes zero. Therefore, the steady state of market equilibrium is given by

$$y^{*}=\left[(1-\tau)-\frac{\tau}{x^{*}}\right]\varphi(x^{*}), \tag{19}$$

$$y^{*}=\frac{1}{\sigma}[(1-\tau)(\sigma+\eta-1)\varphi(x^{*})+\rho], \tag{20}$$

where x^* and y^* mean the steady state values of x and y in the market equilibrium respectively. As can be seen easily, the graph of (18) lies below that of (20) (see figure 1). Moreover, the graph of (15) intersects



Figule. 1 $(\sigma+\eta>1)$

x-axis at the point where $x=\tau/(1-\tau)$. Since the graph of (15) is upward sloping, if (18) takes a positive value when $x=\tau/(1-\tau)$, there exists a unique stationary state (see figure 1). When $\sigma>1$, this condition is always satisfied. Even if $\sigma<1$, when ρ is large enough, then a stationary state exists uniquely. Consequently, we obtain the following propositions:

Proposition 1

Suppose that $\sigma\geq\eta$. Then when σ and ρ is not small, there exists a unique stationary state equilibrium $(x^{*\circ}, y^{*\circ})$.

Proposition 2

The ratio of public capital to private capital at the planned steady-growth path, $x^{*\circ}$, is smaller than that of the steady growth equilibrium where the representative household ignores the effect of public investment on private production, x^{*}, when the rates of income tax are the same level on the both steady growth paths.

Proposition 3

The long-run growth rate of the planned economy is higher than that of the steady growth equilibrium where the household ignores the externalities when the rates of income tax are the same level on both steady-growth paths.

Proof:

The growth rate in the steady state γ is calculated by the following formula:

$$\gamma=\tau\frac{\varphi(x)}{x}.$$

From proposition 2, x^{*} is greater than $x^{*\circ}$; hence, the proposition holds.

Next, we examine the stability of the planned steady-growth path. Linearizing (13), (14) and (12), we obtain:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -(1-\tau)\{\varphi^{*\prime}+\tau\varphi^{*}/(x^{*\circ})^2\}x^{*\circ} & x^{*\circ} & 0 \\ [-(1-\tau)\varphi^{*\prime}\{(1-\tau)/\theta^{*\circ}+\tau\}/\sigma(1-\eta)]y^{*\circ} & y^{*\circ} & \tau(1-\eta)\varphi^{*}y^{*\circ}/\sigma \\ -[(1-\tau)+\theta^{*\circ}\tau]\varphi^{*\prime\prime}/(1-\eta) & 0 & [(1-\tau)+\theta^{*\circ}\tau](1-\eta)\varphi^{*} \end{bmatrix}$$

$$\times \begin{bmatrix} x-x^{*\circ} \\ y-y^{*\circ} \\ \theta-\theta^{*\circ} \end{bmatrix}$$

The determinant of this coefficient matrix $\boldsymbol{M}$ becomes:

$$det\boldsymbol{M} = \mathrm{A}\varphi^{*}(1-\eta)x^{*\circ}y^{*\circ}\left[-\frac{\tau\varphi^{*}(1-\eta)}{(x^{*\circ})^2}+\varphi^{*\prime\prime}\{(1-\tau)x^{*\circ}-\tau\}\right]$$

where $A=[(1-\tau)+\theta^{*\circ}\tau]>0$. Since $x>\tau/(1-\tau)$, $det\boldsymbol{M}$ takes a negative value. Therefore, the coefficient matrix has either one negative eigenvalue and two positive eigenvalues or three negative eigenvalues if all eigenvalues are real numbers. When it has complex eigenvalues, it has one negative real eigenvalue and two conjugate complex eigenvalues. On the other hand, the trace of matrix $\boldsymbol{M}$ becomes;

$$tr\boldsymbol{M} = 2(1-\eta)\left\{(1-\tau)-\frac{\tau}{x^{*\circ}}\right\}\varphi^{*}>0$$

Hence, eigenvalues never become negative simultaneously in the case of real eigenvalues, and when two conjugate eigenvalues, their real parts are positive. Consequently, there is a one-dimentional stable manifold. Since only x is a predetermined variable and the other two variables, y and θ are not predetermined, the planned steady-growth path is stable. Hence, we can state:

Proposition 4

When there is a unique planned steady-growth path, there locally exisis a unique stable path converging to the planned steady-growth path.

3. Effects of Income Taxation

In this section, we consider the effects of a change in the income tax rate on the long-run rate of growth.

Totally differentiating (15) and (18), and calculating $dx^{*\circ}/d\tau$, we get the following :

$$\frac{dx^{*\circ}}{d\tau}=\frac{x^{*\circ}}{(1-\eta)}\,\frac{(1-\eta/\sigma)+(1-\eta)x/\sigma}{(1-\eta/\sigma)\tau+\eta(1-\tau)x/\sigma}>0. \tag{21}$$

Since we assumed that σ is larger than η, the sign of $dx^{*\circ}/d\tau$ is positive. When η is constant, we can calculate $dy^{*\circ}/d\tau$ as follows :

$$\frac{dy^{*\circ}}{d\tau}=\frac{x^{*\circ}y^{*\circ}}{\sigma det\boldsymbol{M}}(x^{*\circ})^{2\eta-2}(\tau-\eta)(\sigma+\eta-1). \tag{22}$$

Therefore, when $\sigma+\eta>1$, the following holds :

$$\frac{dy^{*\circ}}{d\tau}\gtreqqless 0\leftrightarrow\eta\lesseqqgtr\tau.$$

When $\sigma+\eta<1$, the following holds :

$$\frac{dy^{*\circ}}{d\tau}\gtreqqless 0\leftrightarrow\eta\gtreqqless\tau.$$

These results on the sign of $dy^{*\circ}/d\tau$ are the same as those obtained in our previous paper where the representative household ignores the effect of the public investment.

Now we consider the effect of income taxation on the long-run rate of growth. An increase in τ reduces the disposable income (see equation (5)) and thereby reduces investment in private capital ; hence, an increase in τ has a negative effect on the growth rate. However, if the accumulation of private capital decreases, then the ratio of public capital to private capital decreases. This raises the marginal productivity of private capital and has a positive effect on the growth rate. These two opposite effects make the total effect of income taxation on the growth rate a complicated matter. But, when the elasticity of output with respect to public capital η is constant, we can derive a clear result, which is the same as Barro's. The growth rate in the steady state is calculated by using (11) as follows:

$$\gamma^{*\circ}=\tau\frac{\varphi(x^{*\circ})}{x^{*\circ}}.$$

Differentiating this formula with respect to τ we obtain:

$$\frac{d\gamma^{*}}{d\tau}=\left[1-\frac{(1-\eta)\tau}{x^{*\circ}}\frac{dx^{*\circ}}{d\tau}\right]\frac{\varphi(x^{*\circ})}{x^{*\circ}}. \tag{23}$$

By using (21), we can calculate the elasticity of $x^{*\circ}$ with respect to τ as follows:

$$\frac{\tau}{x^{*\circ}}\frac{dx^{*\circ}}{d\tau}=\frac{1}{(1-\eta)}[(1-\eta/\sigma)+(1-\eta)x^{*\circ}/\sigma]$$
$$\times[(1-\eta/\sigma)+\eta(1-\tau)x^{*\circ}/\sigma\tau]^{-1}$$

Let's compare the coefficient of $x^{*\circ}/\sigma$ in the numerator with the one in the denominator. Subtracting the latter value from the former value, we get:

$$(1-\eta)-\frac{(1-\tau)\eta}{\tau}=\frac{\tau-\eta}{\tau}.$$

Therefore, the following relation holds:

$$\frac{d\gamma^{*}}{d\tau}\lesseqqgtr 0\leftrightarrow\frac{(1-\eta)\tau}{x^{*\circ}}\frac{dx^{*\circ}}{d\tau}\gtreqqless 1\leftrightarrow\tau\lesseqqgtr\eta.$$

As a result, we have shown the following proposition in the preceding argument:

Proposition 5

If η is constant, the growth rate of the planned steady state attains its maximum when $\tau=\eta$.

This result is the same as Barro's and our previous one. Even in the model in which the representative households take the accumulation of public capital into account, Barro's result on the maximum growth rate still holds.

4. Concluding Remarks

In this paper, by modifying a simple endogenous growth model constructed by Futagami, Morita and Shibata (1993), we have shown that there is a unique planned steady state and that there is a unique transitional path which converges to the planned steady state. Furthermore, we have examined the effects of income taxation on the steady state and the long-run growth rate, and shown that the comparative static results obtained by Futagami, Morita and Shibata on the steady state equilibrium and the long-run growth rate still hold in the model in which the representative household takes the accumulation of public capital into account, that is, there are no externalities in public capital.

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