Is Liquidity Preference a Behavior Toward Risk? — A Three-Asset Model

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I. Introduction

Generally speaking, two fundamental properties of money have been employed to distinguish it from other assets. One is the use of money as the medium of exchange. The other is its freedom from interest risk. On the basis of the first property, various models of the transactions demand for money have been developed, in one way or another related to the initial formulations of Baumol (1952) and Tobin (1956). From the second property has come the portfolio model, originated by Tobin (1958), which treats the demand for money (narrowly defined) as "behavior towards [interest] risk."

As its title suggests, this paper is devoted to a critical reexamination of the latter source of money demand. It has been inspired by the existence of a large set of close substitutes for money, typically so-called money-market instruments, which can be regarded as riskless, or virtually so, and which pay substantial interest. Intuitively, the availability of many such instruments would appear to make it unlikely that money (again, narrowly defined) would be held in order to cope with interest risk.[1)]

Money-market instruments have received short shrift in the literature on money. Therefore, it behooves us to say something immediately about their various characteristics by way of highlighting their closeness to money with respect to freedom from interest risk.

Some of the money-market instruments are literally riskless. Repurchase agreements (RPs) between banks (and to a lesser extent securities dealers and large savings and loan associations) and nonbank investors are a notable case in point. By their very nature, RPs are devoid of interest risk because these instruments require the borrower to repurchase the securities involved in the loan at their *original* price plus interest. Moreover, over three-quarters of the approximately $ 107 billion of RPs outstanding in the United States in mid-1980 were either overnight in term or callable at any time. Even when maturity dates are fixed, beyond one day, investors unexpectedly needing cash can form *reverse* RPs with the collateral they have taken in, thereby avoiding liquidation risk.

Equally liquid are federal-funds loans to commercial banks by other financial institutions; these are mostly overnight loans with specified interesst rates.

Other money-market instruments, such as Treasury bills, CDs, and bankers' acceptances, do possess positive, if limited, interest risk when sold before maturity. However, these instruments have highly developed secondary markets in which they can be purchased with as short maturities as investors desire and thus are virtually riskless. Also, since 1975, markets for interest-rate futures contracts have been developed that permit investors to hedge (or manage) interrest risk on Treasury bills. Other intruments, such as commercial paper, do not have secondary markets, but their maturity dates are (like those on RPs and CDs) fixed to suit the convenience (cash-need dates) of investors. Moreover, most issuers of commercial paper will normally (though without obligation) redeem this paper before maturity from hard-pressed (regular) investors. And interest-rate futures also exist for commercial paper.

Mention should be made, too, of accounts in money-market (mutual) funds. The assets of these funds consist entirely of the whole array of money-market instruments, plus a tiny amount of demand deposits. If anything, the accounts in these funds are even more liquid than the instruments themselves, because checks may be written on them. In July 1980, these funds aggregated to some \$ 80 billion. The total of all money-market instruments extent at that time in the United States approximated \$ 650 billion.

As noted, the existence of such a rich variety of liquid, money-market instruments and in such large magnitudes, all devoid of interest risk or nearly so and all paying abundant interest, suggests the extreme dubiety of a theory that purports to explain money demand as a response to risk. This skepticism is buttressed by the frantic pace of development of cash-management techniques by business firms. In general, these techniques are designed to reduce perceived variance of cash flows and thus enable investors to reduce their cash holdings and increase the amounts available for investment in money-market instruments.

Doubt and skepticism are not enough, however. They should be supported by argument at least as rigorous as that upon which the original (Tobin) argument rested. To this end, we take up the "risk" demand for money in the context of a three-asset model, one that contains money, a virtually riskless asset, and a risky asset. The model is essentially an extension of Tobin's (1958) formulation. Our purpose is to show that if money-market instruments are explicitly incorporated in a portfolio model, the demand for money as a buffer against interest risk tends to vanish. A model with at least three assets of the types described must be formulated, because some of the (nonmonetary) assets do possess significant interest risk, even while others have little or none, but both types are relevant to the portfolio decision.

Utilizing Tobin's formulation means we are applying Markowitz's (1952) mean-variance model to the analysis of money demand. Although mean-variance analysis is widely accepted, it is not without its critics. A major issue has been the existence of an indifference map between mean and variance, as issue debated by Borch (1969), Feldstein (1969), and Tobin (1969). From this debate we have learned that indifference curves between mean and variance can be drawn only if an investor's preferences can be expressed by a quadratic utility function, or if probability distributions of returns on portfolios are normal. There have been attempts to generalize mean-variance analysis to less restricted framework [e. g., Arrow (1970), Cass and Stiglits (1970, 1972) and Hart (1975)]. However, these attempts have failed to derive clear results in models of many assets.

At the same time, under the restrictive conditions of his model, Tobin's original results stand: For a risk averter, an interior solution is obtained, in which money is a component of an asset portfolio. Nevertheless, for the reason given above, it will be argued here, this result is implausible.

Our program in this paper is as follows: In section 2, we set up a three-asset model along the lines of Tobin (1958), but in the framework of a nonlinear program. As remarked, this requires the use of either the quadratic utility function or the normal distribution of portfolio returns in our model. However, in the face of strong criticism of the quadratic utility function in the present context [for example, Arrow (1970), Hicks (1972)] we are unwilling to stick with the quadratic utility function; so we adopt the normal distribution in our analysis.[2)] Then, in Section 3, the nonlinear program is utilized to note various possible solutions; this procedure is dictated by our interest in the possibility of a corner solution characterized by zero money holdings. In the absence of the quadratic utility function, results are derived without explicit specification of a utility function. In Section 4, a utility function is introduced that corresponds to constant absolute risk aversion. Demand functions for each of the three assets are then derived and evaluated under the condition that the risk of one of the interest-bearing assets is virtually nonexistent. In Section 5, a similar analysis is conducted based on a quadratic utility function. In the process, we examine the implications of this form of the utility function for portfolio analysis in general and for money demand in particular. Finally, in Section 6, we summarize our results and state possible extentions. In an appendix we present some comparative-static results. Although these are tangential to our main interest, we shall have occasion to mention their properties in the text; so it may be convenient to have them for reference.

II. Models and Assumptions

As noted, our model follows in the footsteps of Tobin's model (1958), except for the introduction of a third, almost riskless, asset.[3] For convenience, we label our three assets money (narrowly defined), a short-term asset, and a long-term asset, respectively.

In our model, the investor's behavior is treated as the outcome of an expected utility maximization for the choice of asset holdings, but at the outset we do not specify an expected utility function. Basic to this investment analysis is the risk-return relation and its estimated distribution by the investor.

[Assumption 1]

If we express an interest rate for each asset as r_i ($i=1, 2, 3$), they have the following properties, $r_1=0$, $r_2>0$, and $r_3>0$.

Next, we express the capital gain or loss of each asset by the random variables g_i ($i=1, 2, 3$).

[Assumption 2]

Money has no capital gain or loss, i. e., $g_1=0$. Hence, g_1 is actually not a random variable.

[Assumption 3]

The investor's estimated distribution function for capital gains or losses has a joint normal distribution; the means are zero, the variances are σ_2^2, σ_3^2, and the covariance is $\sigma_{23}=\rho_{23}\sigma_2\sigma_3$. The coefficient ρ_{23} is the correlation coefficient between g_2 and g_3, and we assume $-1<\rho_{23}<1$.

Thus, the measure of return on each asset is the expected value of its interest rate, and the measure of risk of each asset is the standard deviation of capital gain or loss: σ_i $(i=2, 3)$.

[Assumption 4]

The short-term asset is less risky than the long-term asset, i. e. $\sigma_2<\sigma_3$. By assumption 2, $g_1=0$; so that σ_1 must be zero. Hence, in our model, money has no risk.

In this paper we discuss only the share of each asset, instead of the absolute size of demand. Therefore, we implicitly assume the share of each asset to be independent of the amount of initial wealth.

[Assumption 5]

The shares of each asset A_i $(i=1, 2, 3)$ add to one, and the A_i' s are non-negative.

Given the foregoing (five) assumptions, we can express the estimated return R from one unit of a portfolio as follows:

$$R\equiv A_2(r_2+g_2)+A_3(r_3+g_3) \tag{1}$$

From Assumption 3, R must follow a nomal distribution with mean μ_R, and variance σ_R^2. These parameters are calculated as follows:

$$\mu_R\equiv E[R]=A_2r_2+A_3r_3 \tag{2}$$

$$\sigma_R^2\equiv V(R)=E[\{R-E[R]\}^2]=A_2^2\sigma_2^2+2A_2A_3\sigma_{23}+A_3^2\sigma_3^2 \tag{3}$$

Now we can specify our maximization problem. Because we are principally concerned with the possibility of a corner solution featuring zero money balances, we must set up a nonlinear programming model. Accordingly, our problem is:

$$\text{Max } EU=\text{Max } E[U(R)]=\text{Max } EU(\mu_R, \sigma_R^2) \tag{4}$$

$$\begin{aligned}\text{S.t. } & \mu_R=A_2r_2+A_3r_3\\ & \sigma_R^2=A_2^2\sigma_2^2+2A_2A_3\sigma_{23}+A_3^2\sigma_3^2\\ & A_1+A_2+A_3=1\\ & A_1, A_2, A_3\geqq 0\end{aligned}$$

Not that the equality in our objective function need not hold. However, Chipman[4] (1973) has derived the necessary and sufficient conditions on the utility function to guarantee this equality when the distribution of returns is normal. Hence, we shall proceed as if these conditions are satisfied.[5]

Our final assumption relates to the signs of derivatives of our utility function.

[Assumption 6]

$U(R)$ has a positive first-order derivative and negative second-ordor derivative. The investor is thus a risk averter.

This assumption implies that the utility function is concave. Hence, EU is concave, strictly decreasing in σ_R^2. By the same token, if U is strictly increasing in R, EU is strictly increasing in μ_R. [Proofs are given in Pratt, *et al.* (1965)].

Having set up our problem, we can proceed to the general solutions.

III. General Solutions

In this section, we shall solve the Lagrangean problem formulated from (4). Ultimately, our interest is the evaluation of the demand functions in a quantitative fashion. For that, however, we must have a specific unility function. But, first, let us derive the Kuhn-Tucker conditions based on a general utility function. This will give us a unified view for the analysis that follows.

The Lagrangean function (L) is:[6]

$$L = EU(A_2 r_2 + A_3 r_3,\ A_2^2\sigma_2^2 + 2A_2A_3\sigma_{23} + A_3^2\sigma_3^2) + \lambda(1 - A_1 - A_2 - A_3) \tag{5}$$

In turn, the Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial A_1} = -\lambda \leqq 0 \tag{6}$$

$$\frac{\partial L}{\partial A_2} = \frac{\partial EU}{\partial \mu_R}\frac{\partial \mu_R}{\partial A_2} + \frac{\partial EU}{\partial \sigma_R^2}\frac{\partial \sigma_R^2}{\partial A_2} - \lambda$$

$$=r_2\cdot\frac{\partial EU}{\partial\mu_R}+2(A_2\sigma_2^2+A_3\sigma_{23})\frac{\partial EU}{\partial\sigma_R^2}-\lambda\leqq 0 \tag{7}$$

$$\frac{\partial L}{\partial A_3}=\frac{\partial EU}{\partial\mu_R}\frac{\partial\mu_R}{\partial A_3}+\frac{\partial EU}{\partial\sigma_R^2}\frac{\partial\sigma_R^2}{\partial A_3}-\lambda$$

$$=r_3\cdot\frac{\partial EU}{\partial\mu_R}+2(A_2\sigma_{23}+A_3\sigma_3^2)\frac{\partial EU}{\partial\sigma_R^2}-\lambda\leqq 0 \tag{8}$$

$$\frac{\partial L}{\partial A_1}A_1+\frac{\partial L}{\partial A_2}A_2+\frac{\partial L}{\partial A_3}A_3=0 \tag{9}$$

$$\frac{\partial L}{\partial\lambda}=1-A_1-A_2-A_3=0 \tag{10}$$

$$\left.\begin{array}{l}A_1\geqq 0\\ A_2\geqq 0\\ A_3\geqq 0\end{array}\right\} \tag{11}$$

If we take into account all possible solutions, corner and interior, there are seven possible cases: $A_1=A_2=A_3=0$; $A_1=A_3=0$, $A_2=1$; $A_1=A_2=0$, $A_3=1$; $A_1\neq 0$, $A_2\neq 0$, $A_3=0$; $A_1\neq 0$, $A_2=0$, $A_3\neq 0$; $A_1=0$, $A_2\neq 0$, $A_3\neq 0$; $A_1\neq 0$, $A_2\neq 0$, $A_3\neq 0$. Each of these solutions has a corresponding set of first-order conditions. However, this study is motivated by the question of whether or not money is held as a response to interest risk even in the presence of virtually riskless money-market instruments (i. e., even when σ_2 is very small); therefore, we shall limit the discussion to the last two cases.

[Case i] $A_1=0$, $A_2\neq 0$, $A_3\neq 0$.

This case requires that

$$\frac{\partial L}{\partial A_1}<0,\ \frac{\partial L}{\partial A_2}=0,\ \frac{\partial L}{\partial A_3}=0.$$

By (7) and (8), we have

$$\frac{\partial L}{\partial A_2}=\frac{\partial EU}{\partial\mu_R}\cdot r_2+2(A_2\sigma_2^2+A_3\sigma_{23})\frac{\partial EU}{\partial\sigma_R^2}-\lambda=0$$

$$\frac{\partial L}{\partial A_3}=\frac{\partial EU}{\partial\mu_R}\cdot r_3+2(A_2\sigma_{23}+A_3\sigma_3^2)\frac{\partial EU}{\partial\sigma_R^2}-\lambda=0$$

Therefore,

$$(r_2-r_3)\frac{\partial EU}{\partial \mu_R}+2(A_2\sigma_2^2+A_3\sigma_{23}-A_2\sigma_{23}-A_3\sigma_3^2)\frac{\partial EU}{\partial \sigma_R^2}=0$$

By (6), $A_2=1-A_3$. So,

$$(r_2-r_3)\frac{\partial EU}{\partial \mu_R}+2\{(1-A_3)\sigma_2^2+A_3\sigma_{23}-(1-A_3)\sigma_{23}-A_3\sigma_3^2\}\frac{\partial EU}{\partial \sigma_R^2}=0$$

Hence,

$$(r_2-r_3)\frac{\partial EU}{\partial \mu_R}=2\{(\sigma_2^2-\sigma_{23})-(\sigma_2^2-2\sigma_{23}+\sigma_3^2)A_3\}\frac{\partial EU}{\partial \sigma_R^2}=0$$

Thus

$$A_3=\frac{(r_2-r_3)\dfrac{\partial EU}{\partial \mu_R}+2(\sigma_2^2-\sigma_{23})\dfrac{\partial EU}{\partial \sigma_R^2}}{2(\sigma_2^2-2\sigma_{23}+\sigma_3^2)\dfrac{\partial EU}{\partial \sigma_R^2}} \tag{11}$$

and

$$A_2=1-A_3. \tag{12}$$

[Case ii] $A_1\neq 0$, $A_2\neq 0$, $A_3\neq 0$.

This case requires that

$$\frac{\partial L}{\partial A_1}=\frac{\partial L}{\partial A_2}=\frac{\partial L}{\partial A_3}=0.$$

By (6) $\lambda=0$; so

$$\frac{\partial L}{\partial A_2}=\frac{\partial EU}{\partial \mu_R}\cdot r_2+2(A_2\sigma_2^2+A_3\sigma_{23})\frac{\partial EU}{\partial \sigma_R^2}=0$$

$$\frac{\partial L}{\partial A_3}=\frac{\partial EU}{\partial \mu_R}\cdot r_3+2(A_2\sigma_{23}+A_3\sigma_3^2)\frac{\partial EU}{\partial \sigma_R^2}=0$$

Therefore,

$$\frac{\partial EU}{\partial \mu_R}\Big/\frac{\partial EU}{\partial \sigma_R^2}=-\frac{2(A_2\sigma_2^2+A_3\sigma_{23})}{r_2}=-\frac{2(A_2\sigma_{23}+A_3\sigma_3^2)}{r_3} \tag{13}$$

so that

$$A_2=\frac{r_2\sigma_3^2-r_3\sigma_{23}}{r_3\sigma_2^2-r_2\sigma_{23}}\cdot A_3 \tag{14}$$

From (14) we can see that the division of the portfolio between the short-term and long-term assets is independent of the share of money.

Furthermore,

$$\frac{\partial EU}{\partial \mu_R}\cdot r_2+2\left\{\frac{(r_2\sigma_3^2-r_3\rho_{23}\sigma_2\sigma_3^2)\sigma_2^2}{r_3\sigma_2^2-r_2\sigma_{23}}+\sigma_{23}\right\}\frac{\partial EU}{\partial \sigma_R^2}\cdot A_3=0$$

which implies that

$$A_3=\frac{(r_3\sigma_2^2-r_2\sigma_{23})\left(\dfrac{\partial EU}{\partial \mu_R}\right)}{2(1-\rho_{23}^2)\sigma_2\sigma_3^2\left(\dfrac{\partial EU}{\partial \sigma_R^2}\right)} \tag{15}$$

$$A_2=\frac{(r_2\sigma_3^2-r_3\sigma_{23})\left(\dfrac{\partial EU}{\partial \mu_R}\right)}{2(1-\rho_{23}^2)\sigma_2^2\sigma_3\left(\dfrac{\partial EU}{\partial \sigma_R^2}\right)} \tag{16}$$

$$A_1=1-(A_2+A_3) \tag{17}$$

With these results, we have taken the solution of our Lagrangean problem as far as we can without specifying the utility function. We must now determine whether Case i) or Case ii) will prevail in the world of reality. If Case ii) prevails, there will be some "risk" demand for money; if Case i) prevails, on the other hand, there will be none.

In the following two sections, we introduce two specific forms of the utility function. Then we examine the above question in the context of the almost riskless short-term asset.

IV. Money Demand with a Utility Function Characterized by Constant, Absolute Risk Aversion

In this section we derive the demand function for money with the aid of the utility function corresponding to constant absolute risk aversion, first set forth by Pratt (1964). Although this is a restricted class of utility functions, it has the advantage of satisfying all the requirements[7] imposed by Arrow (1970) for a utility function appropriate to the analysis of economic behavior under uncertainty.

The utility function has the following form:

$$U(R)=-e^{-kR},\ k>0,\ U'(R)>0,\ U''(R)<0 \tag{18}$$

The corresponding expected utility function is

$$EU(\mu_R,\ \sigma_R)=-e^{-k(\mu_R-\frac{1}{2}k\sigma_R^2)} \tag{19}$$

The first-order patirials of (19) are;

$$\frac{\partial EU}{\partial \mu_R}=ke^{-k(\mu_R-\frac{1}{2}k\sigma_R^2)}>0 \tag{20}$$

$$\frac{\partial EU}{\partial \sigma_R^2}=-\frac{1}{2}k^2e^{-k(\mu_R-\frac{1}{2}k\sigma_R^2)}<0 \tag{21}$$

Thus [together with (18)] we obtain concave-upward, positively sloped indifference curves between μ_R and σ_R^2, without any restriction over the range.

We now consider the problem of deriving a money-demand function when we apply this utility function to our three-asset model. Following the procedure of section III, we specify the Lagrangean function as:

$$L=-e^{-k\{(A_2r_2+A_3r_3)-\frac{1}{2}k(A_2^2\sigma_2^2+2A_2A_3\sigma_{23}+A_3^2)\}}+\lambda(1-A_1-A_2-A_3) \tag{22}$$

Now, we can derive the Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial A_1}=-\lambda\leqq 0 \tag{23}$$

$$\frac{\partial L}{\partial A_2}=k\{r_2-k(A_2\sigma_2^2+A_3\sigma_{23})\}e^{-k(\mu_R-\frac{1}{2}k\sigma_R^2)}-\lambda\leqq 0 \tag{24}$$

$$\frac{\partial L}{\partial A_3}=k\{r_3-k(A_2\sigma_{23}+A_3\sigma_3^2)\}e^{-k(\mu_R-\frac{1}{2}k\sigma_R^2)}-\lambda\leqq 0 \tag{25}$$

$$\frac{\partial L}{\partial A_1}A_1+\frac{\partial L}{\partial A_2}A_2+\frac{\partial L}{\partial A_3}A_3=0 \tag{26}$$

$$\frac{\partial L}{\partial A}=1-A_1-A_2-A_3=0 \tag{27}$$

$$\left.\begin{array}{l} A_1\geqq 0 \\ A_2\geqq 0 \\ A_3\geqq 0 \end{array}\right\} \tag{28}$$

This system can be solved according to the two cases set forth in Section III. On the basis of the results obtained there, these solutions are:

[Solution for Case i)]

By (11), A_3 can be expressed as

$$A_3=\frac{(r_2-r_3)\dfrac{\partial EU}{\partial \mu_R}+2(\sigma_2^2-\sigma_{23})\dfrac{\partial EU}{\partial \sigma_R^2}}{2(\sigma_2^2-2\sigma_{23}+\sigma_3^2)\dfrac{\partial EU}{\partial \sigma_R^2}} \tag{29}$$

But, by (20) and (21), we know that the ratio $\dfrac{\partial EU}{\partial \mu_R}\Big/\dfrac{\partial EU}{\partial \sigma_R^2}$ is independent of the level of the A_i's; so by substitution,

$$A_3=\frac{-(r_2-r_3)+k(\sigma_2^2-\sigma_{23})}{k\alpha} \tag{30}$$

where $\alpha\equiv\sigma_2^2-2\sigma_{23}+\sigma_3^2$.

Then we can derive A_2 by

$$A_2=1-A_3=\frac{(r_2-r_3)-k(\sigma_{23}-\sigma_3^2)}{k\alpha} \tag{31}$$

[Solution for Case ii)]

By (15), (16), (20), and (21), we can derive A_2 and A_3 in a manner paralleling Case i):

$$A_2=\frac{r_2\sigma_3-r_3\rho_{23}\sigma_2}{k\sigma_2^2\sigma_3(1-\rho_{23}^2)} \tag{32}$$

$$A_3=\frac{r_3\sigma_2-r_2\rho_{23}\sigma_3}{k\sigma_2\sigma_3^2(1-\rho_{23}^2)} \tag{33}$$

The derivation of A_1 follows:

$$A_1=1-A_2-A_3=1-\frac{r_2(\sigma_3^2-\sigma_{23})+r_3(\sigma_2^2-\sigma_{23})}{k\sigma_2^2\sigma_3^2(1-\rho_{23}^2)} \tag{34}$$

Now, we can consider which of these two cases is likely to prevailing in the world in which the short-term asset possesses negligible risk. To do this, we analyze the behavior of money demand with respect to r_2. Because the less risky, short-term asset is much the closer substitute for money, we regard its yield as the key factor in the specification of money demand. First of all, on the basis of (34), we derive the range for r_2 within which money holdings occur. For A_1 to be positive, the following relation must hold:

$$1+\frac{r_2(\sigma_{23}-\sigma_3^2)+r_3(\sigma_{23}-\sigma_2^2)}{k\sigma_2^2\sigma_3^2(1-\rho_{23}^2)}>0$$

Solving this relation for r_2, we get:

$$r_2<\frac{k\sigma_2^2\sigma_3^2(1-\rho_{23}^2)-r_3(\sigma_2^2-\sigma_{23})}{\sigma_3^2-\sigma_{23}}\equiv l_1 \tag{35}$$

Similarly, for $A_1<1$, we have,[8]

$$r_2>\frac{\sigma_{23}-\sigma_2^2}{\sigma_3^2-\sigma_{23}}\cdot r_3\equiv l_2 \tag{36}$$

Hance given expected risk and the interest rate on the long-term asset, if r_2 is greater than l_1, there are no money holdings; but if r_2 lies between l_1 and l_2, Case (ii) prevails, and money is a component of a risk averter's portfolio. Figure 1 shows these relations.[9]

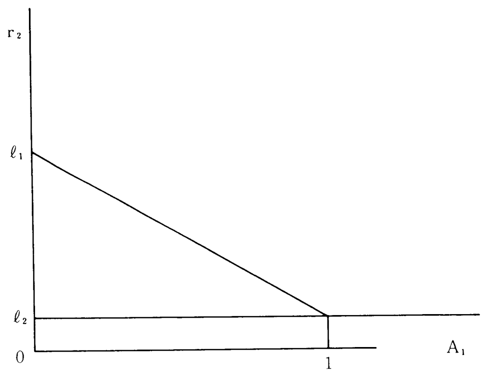

Figure 1

Evaluation values of those critical values of l_1 and l_2 is an empirical matter. So let us take the case in which σ_2 tends to zero. Remembering that $\sigma_{23}=\rho_{23}\sigma_1\sigma_2=0$, we can see that, in this circumstance,

$$\lim_{\sigma_2\to 0} l_1=0, \text{ and } \lim_{\sigma_2\to 0} l_2=0.$$

This means that if there is really a riskless asset besides money, there is no possibility that money will be held as a protection againet interest risk. And, as we have seen, such assets do exist.

To strengthen our argument, we can show that the range between l_1 and l_2 becomes smaller as σ_2 becomes smaller. Since

$$l_1 - l_2 = \frac{k\sigma_2^2\sigma_3^2(1-\rho_{23}^2)}{\sigma_3^2 - \sigma_{23}}$$

$$\frac{\partial(l_1 - l_2)}{\partial\sigma_2} = \frac{2k\sigma_2\sigma_3^2(1-\rho_{23}^2)(\sigma_3^2-\sigma_{23}) + k\rho_{23}\sigma_2^2\sigma_3^3(1-\rho_{23}^2)}{(\sigma_3^2-\sigma_{23})^2} > 0$$

Hence, if the risk of the short-term asset is very small (and it pays enough interest to cover transaction costs), there appears to be no reason to hold money in order to avoid risk. The portfolio-selection decision is dominated by Case i), and as σ_2 tends to zero, the optimal portfolio is characterized by[10]:

$$A_1 = 0\ ;\ A_2 = \frac{(r_2 - r_3) + k\sigma_3^2}{k\sigma_3^2}\ ;\ A_3 = \frac{-(r_2 - r_3)}{k\sigma_3^2}.$$

V. Money Demand with a Quadratic Utility Function

The results of Section IV were obtained with a utility function marked constant, absolute risk aversion. In this section, we deploy the quadratic utility function, utilized in Tobin's original analysis, to investigate the portfolio decision. Our principal purpose is to determine the extent to which Tobin's results depend on this specific form of the utility function. If the latter is crucial, it is important to know this, because the quadratic utility function suffers from the major defect of implying that all risky assets are inferior goods [See Arrow (1970) and Hicks(1972)].

First, we write the quadratic utility function as follows:

$$U(R) = (1+b)R + bR^2 \tag{37}$$

Since we are considering only the behavior of a risk averter, b should be negative; hence we restrict b to the interval $-1<b<0$. One problem with the quadratic utility function is that this restriction on b is not sufficient to satisfy Assumption 6. We can see this easily be differentiating (37) with respect to R:

$$U'(R)=(1+b)+2bR \tag{38}$$

The right-hand side of (38) need not be positive. Hence, we must impose the additional restriction that $R<-\left(\frac{1+b}{2b}\right)$ to guarantee its positivity. Bearing this in mind, note, further, that differentiation of (38) implies that $U''(R)<0$. The expected utility function now takes the following form:

$$EU(\mu_R,\ \sigma_R)=(1+b)\mu_R+b(\mu_R^2+\sigma_R^2) \tag{39}$$

The first-order partials for this expected utility function are:

$$\frac{\partial EU}{\partial\mu_R}=(1+b)+2b\mu_R>0 \tag{40}$$

$$\frac{\partial EU}{\partial\sigma_R^2}=b<0 \tag{41}$$

The sign of $\frac{\partial EU}{\partial\mu_R}$ may not be evident, but if $\mu_R<-\frac{1+b}{2b}$, the sign must be positive. Since we have required all R to be less than $-\left(\frac{1+b}{2b}\right)$, the expected value of R should satisfy this condition. Hence, under this restriction, the indifference curves between μ_R and σ_R^2 are positive in slope and (because of the concavity of U) concave upward.

Now we may set up the Lagrangean function:

$$L=(1+b)(A_2r_2+A_3r_3)+b\{(A_2r_2+A_3r_3)^2+(A_2^2\sigma_2^2+2A_2A_3\sigma_{23}+A_3^2\sigma_3^2)\}$$
$$+\lambda(1-A_1-A_2-A_3) \tag{42}$$

for which the Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial A_1}=-\lambda\leqq 0 \tag{43}$$

$$\frac{\partial L}{\partial A_2}=\{(1+b)+2b\mu_R\}r_2+2b(A_2\sigma_2^2+A_3\sigma_{23})-\lambda\leqq 0 \tag{44}$$

$$\frac{\partial L}{\partial A_3}=\{(1+b)+2b\mu_R\}r_3+2b(A_2\sigma_{23}+A_3\sigma_3^2)-\lambda\leqq 0 \tag{45}$$

$$\frac{\partial L}{\partial A_1}A_1+\frac{\partial L}{\partial A_2}A_2+\frac{\partial L}{\partial A_3}A_3=0 \tag{46}$$

$$\frac{\partial L}{\partial \lambda}=1-A_1-A_2-A_3\geqq 0 \tag{47}$$

$$\left.\begin{array}{l} A_1\geqq 0\\ A_2\geqq 0\\ A_3\geqq 0\end{array}\right\} \tag{48}$$

As we did in Section Ⅳ, we solve this system for Cass i) and ii) in Section Ⅲ.

[Solution for Case i)]

In contrast with the utility function previously employed, with the quadratic utility function $\frac{\partial EU}{\partial \mu_R}\Big/\frac{\partial EU}{\partial \sigma_R^2}$ is not independent of A_2, A_3, therefore, we cannot use(11) and (12) directly. But, by (44), (45), and (46), we can derive

$$\begin{aligned}\lambda&=\{(1+b)+2b\mu_R\}r_2+2b(A_2\sigma_2^2+A_3\sigma_{23})\\&=\{(1+b)+2b\mu_R\}r_3+2b(A_2\sigma_{23}+A_3\sigma_3^2)\end{aligned} \tag{49}$$

Further, under Case i), $A_2+A_3=1$. Hence, the solutions for A_2 and A_3 are:

$$A_2=-\frac{[(1+b)(r_2-r_3)+2b\{r_3(r_2-r_3)+\sigma_{23}-\sigma_3^2\}]}{2b[(r_2-r_3)^2+\alpha]} \tag{50}$$

$$A_3=-\frac{[(1+b)(r_2-r_3)+2b\{r_2(r_2-r_3)+\sigma_2^2-\sigma_{23}\}]}{2b[(r_2-r_3)^2+\alpha]} \tag{51}$$

[Solution for Case ii)]

By (43)—(46), we have

$$\{(1+b)+2b\mu_R\}r_2+2b(A_2\sigma_2^2+A_3\sigma_{23})=0$$

$$\{(1+b)+2b\mu_R\}r_3+2b(A_2\sigma_{23}+A_3\sigma_3^2)=0$$

Therefore,

$$A_2=-\frac{\{(1+b)+2b\mu_R\}r_2+2b\sigma_{23}A_3}{2b\sigma_2^2}=\frac{\{(1+b)+2b\mu_R\}r_3+2b\sigma_{23}A_3}{2b\sigma_{23}}$$

Hence, the solution for A_3 is:

$$A_3=-\frac{(1+b)}{2b}\cdot\frac{r_3\sigma_2^2-r_2\sigma_{23}}{\Delta} \tag{52}$$

And by substitution, we can derive A_2:

$$A_2=-\frac{(1+b)}{2b}\cdot\frac{r_2\sigma_3^2-r_3\sigma_{23}}{\Delta} \tag{53}$$

Since $A_1=1-A_2-A_3$, we get,

$$A_1=1+\frac{(1+b)}{2b}\cdot\frac{\beta}{\Delta} \tag{54}$$

where $\Delta\equiv\gamma+(1-\rho_{23}^2)\sigma_2^2\sigma_3^2$

$$\gamma\equiv r_2^2\sigma_3^2-2r_2r_3\rho_{23}\sigma_2\sigma_3+r_3^2\sigma_2^2$$
$$=(r_2\sigma_3-r_3\rho_{23}\sigma_2)^2+r_3^2\sigma_2^2(1-\rho_{23}^2)>0$$

and $\beta\equiv r_2(\sigma_3^2-\sigma_{23})+r_3(\sigma_2^2-\sigma_{23})$

Now, with these solutions before us, we may proceed as we did in Section IV. Let us consider the demand function for money. By (54) the condition for A_1 to be less than one is:

$$r_2\leqq\frac{\sigma_2^2-\sigma_{23}}{\sigma_{23}-\sigma_3^2}\cdot r_3=l_2 \tag{55}$$

However, deriving the condition $A_1>0$ is somewhat complicated, because it entails solving the following quadratic equation:

$$2b\sigma_3^2r_2^2=\{(1+b)(\sigma_3^2-\sigma_{23})-4br_3\sigma_{23}\}\sigma_3r_2$$
$$+2b\{r_3^2\sigma_2^2+(1-\rho_{23}^2)\sigma_2^2\sigma_3^2\}+(1+b)(\sigma_2^2-\sigma_{23})r_3>0 \tag{56}$$

But if we derive the range of r_2 that satisfies the above inequality, we obtain:

$$l_1^{\mathrm{L}}<r_2<l_1^{\mathrm{U}} \tag{57}$$

where

$$l_1^{\mathrm{L}}\equiv\frac{-1}{4b\sigma_3^2}[\{(1+b)(\sigma_3^2-\sigma_{23})-4br_3\sigma_{23}\}-\{\{(1+b)(\sigma_3^2-\sigma_{23})-4br_3\sigma_{23}\}^2$$
$$-\{16b^2\sigma_3^2[r_3^2\sigma_2^2+(1-\rho_{23}^2)\sigma_2^2\sigma_3^2]-8b(1+b)(\sigma_2^2-\sigma_{23})r_3\sigma_3^2\}\}^{\frac{1}{2}}].$$

and

$$l_1^U \equiv \frac{-1}{4b\sigma_3^2}[\{(1+b)(\sigma_3^2-\sigma_{23})-4br_3\sigma_{23}\}+\{\{(1+b)(\sigma_3^2-\sigma_{23})-4br_3\sigma_{23}\}^2$$
$$-\{16b^2\sigma_3^2[r_3^2\sigma_2^2+(1-\rho_{23}^2)\sigma_2^2\sigma_3^2]-8b(1+b)(\sigma_2^2-\sigma_{23})r_3\sigma_3^2\}\}^{\frac{1}{2}}].$$

The graph between A_1 and r_2 based upon (57) is drawn in Figure 2.

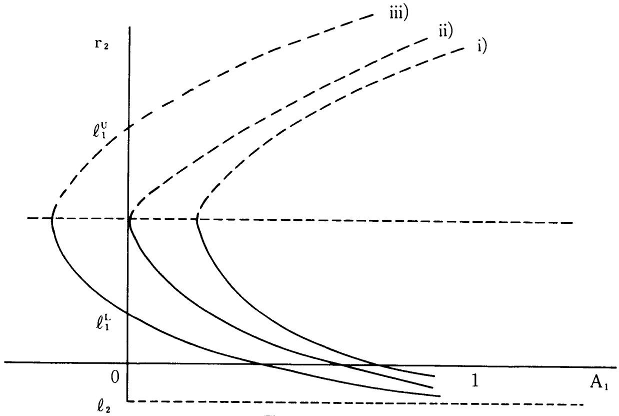

Figure 2

In this graph, curve i) results when the sign of the expression within the square root is negative, i. e., the root is imaginary; curve ii) results when there are multiple roots; curve iii) is the consequence of two real roots. Unless $\sigma_2 > \rho_{23}\sigma_3$, we cannot exclude the possibility of curve i), viz., the possibility that A_1 is always positive.

Since (57) is too complicated to evaluate generally, let us see what happens when σ^2 tends to zero. Under this condition, $l_1^U \to \frac{-(1+b)}{2b}$ and $l_1^L \to 0$. So the curve relating A_1 to r_2, now looks like that in Figure 3.

Thus, if we consider only the range of r_2 associated with the negatively sloped demand curve for A_1, we can see that no matter how small r_2 is, a very small σ_2 implies the absence of money in a risk-averter's optimum portfolio.

However, an odd thing about this result is that an economically meaningful solution, characterized by $\partial A_1/\partial r_2 < 0$, exists only when r_2 is less then $-(1+b)/4b$.[11] We cannot say how restrictive this condition is,

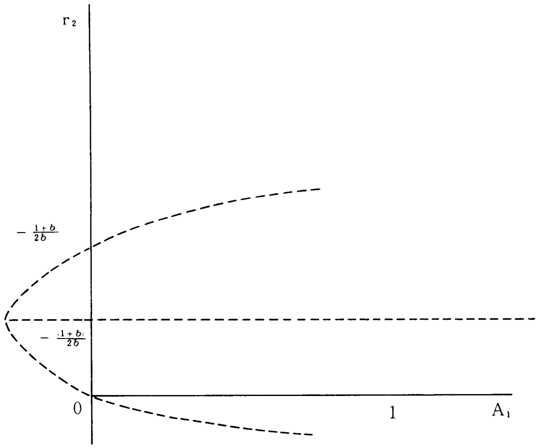

Figure 3

but this fact must give us pause in using the quadratic utility function in a portfolio analysis of money demand. Furthermore, if we revert to Equation (54) and find the limit of A_1 as $\sigma_2 \to 0$ we don't get $A_1=0$; instead, we get

$$\lim_{\sigma_2 \to 0} A_1 = 1 + \frac{1+b}{2b}\frac{1}{r_2} \tag{58}$$

for which

$$\frac{dA_1}{dr_2} > 0 \ \left(\text{because } \frac{1+b}{2b} < 0\right).$$

In light of all the foregoing results, it would appear that even under a quadratic utility function, portfolio therry *per se* fails to provide an explanation for money demand in an economy in which virtually riskless, short-term credit instruments abound. As earlier, the portfolio decision is dominated by Case i), for which, as $\sigma_2 \to 0$, the optimal portfolio has $A_1 = 0$ and

$$A_2 = -\frac{1}{2b[\sigma_3^2 + (r_2 - r_3)^2]}[(1+b)(r_2 - r_3) + 2b\{r_3(r_2 - r_3) - \sigma_3^2\}]$$

$$A_3 = -\frac{1}{2b[\sigma_3^2 + (r_2 - r_3)^2]}[(1+b)(r_2 - r_3) + 2b\{r_2(r_2 - r_3)\}].$$

VI. Concluding Remarks

In this paper, we have established that under the Tobin framework, if there is a virtually riskless asset which pays interest, there is no reason for a risk averter to include money in his portfolio as a means of coping with interest (or market) risk.

Several extentions are possible. One is to relax the implicit assumption that the share of each asset in a portfolio is independent of the initial stock of wealth. Even in this case either a quadratic utility function or a normal distribution must be assumed. [See Maysher (1978), Feldstein (1978)]. Another extension would involve the introduction of inflationary expectations into our model, as Boonkamp (1978) did for the two-asset case. Additionally, the demand for precautionary balances, held by investors facing uncertain cash needs [see Tsiang (1969)], might be examined.

However, in all there cases, the basic economic logic appears to be the same: Various money-market instruments dominate money in an investor's portfolio. This leads us to the conviction that key to the holding of money is the need to make payments on assorted transactions — that the demand for money is fundamentally a *transactions* demand.

Footnotes

1) Here and there in the literature on monetary theory, others have expressed similar discomfiture with the Tobin theory of money demand — by pointing to the existence of savings and time deposits, assets that possess the same risk properties as money and pay interest; see, e. g., Barro and Fischer (1976). Although correct, there is something naieve about this allusion to savings and time deposits. It is true that both categories of deposits are free of interest risk in the conventional sense of capital loss. But time deposits are still exposed to a kind of interest risk, because they can be liquidated before maturity only with interest penalty. More impor-

tant, business firms are either denied access to savings deposite or, as in the United States since late 1975, can hold a maximum of $ 150,000 (per account) in such deposits at commercial banks, a restriction that effectively precludes large firms from holding them. At the same time, although business firms can own most time deposits, they typically do not,; they are loathe to tie up funds in the long-term maturities, and they can usually obtain the same or higher yields on other types of short-term debt instrumens that are also negotiable. In the United States, households have long accounted for about one-third of demand deposits, business firms owning most of the rest. So any effort to rest a case against the Tobin theory of money demand on the existence of savings- and time-deposit substitutes clearly gets at only small part of the problem. This stricture extends to so-called NOW and ATS accounts, resently made available in the United States; these interest-bearing demand deposits (disguised under other names) are also denied to business firms.

2) There is some empirical support for the assumption of normality in the distribution of portfolio returns. See Fama (1976), pp. 43–44.

3) Until we reach the possibility of a corner solution, consideration of a riskless third asset is deferred in faver of one whose risk is intermediate between the risks of the other two assets.

4) See Theorem 1 (necessary condition), and Theorem 2 (sufficient condition), in this paper.

5) When we solve the problem explicitly, we shall use the kind of utility function that satisfies the conditions for equality.

6) Since the constraints for μ_R and σ_R^2 are equality constraints, we can substitute them into our objective function.

7) These requirements are:

$U'(R)>0$ (non-satiation)

$U''(R)<0$ (risk aversion)

$$\frac{d\left(-\dfrac{U''(R)}{U'(R)}\right)}{dR} \leqq 0 \quad \text{(decreasing absolute risk aversion)}$$

$$\frac{d\left(-\dfrac{RU''(R)}{U(R)}\right)}{dR} \geqq 0 \quad \text{(increasing relative risk aversion)}$$

These conditions basically imply that a risk-averse investor would hold an absolutely larger amount, but a smaller proportion, of risky assets when his return increases.

8) This value may not be positive, but that only means that even though r_2 is zero, A_1 may not be 1.

9) To check the slope, see the appendix on the comparative statics.

10) Note, also, that if $r_3 > r_2$, $A_3 > 0$. Further, if $r_3 - r_2 > k\sigma_3^2$, $A_2 = 0$ and $A_3 = 1$; this result may be interpreted to mean that in the event of a negatively sloped yield curve, an investor may choose to hold only long-term assets, provided the excess of the long-term over the short-term interest rate is large enough to offset the risk of long-term assets. Finally, if $r_3 < r_2$, then $A_3 = 0$ and $A_2 = 1$, a result that makes sense under the characteristic of Case i), *viz.*, $\sigma_2^2 \to 0$.

11) This point is related to Tobin's (1958) condition that $\sigma_2 > r_2$ to assure a meaningful solution. Actually, the two conditions are (roughly) equivalent, because if we take the limit as $\sigma_2 \to 0$ in (52), $A_3 = 0$; thus, under this condition, we really have just two assets.

Appendix : Some Comparative Static Results.

In this appendix, we summarize some of the comparative-static results that derive from our demand functions for assets. Although these results are tangential to our principal concerns in this paper, on several occasions we have alluded to tham. This was especially true in Section V, where it was necessary to take into account the comparative-static results in order to evaluate the money-demand function. So it might be helpful to summarize some of the results in one place.

First of all, we expect the following sign patterne for asset demands:

$$\frac{\partial A_i}{\partial r_j} \leqq 0 \text{ when } i \neq j, \quad \frac{\partial A_i}{\partial r_i} \geqq 0,$$

$$\frac{\partial A_i}{\partial \sigma_j} \geqq 0 \text{ when } i \neq j, \quad \frac{\partial A_i}{\partial \sigma_i} \leqq 0.$$

Now, we shall check the conditions necessary to satisfy these sign patterns under Cases i) and ii) in both Sections IV and V; in addition, we shall derive them for the asset-demand functions when σ_2 tends to zero. [For obvious reasons, σ_{23} $(= \rho_{23}\sigma_2\sigma_3)$ is spelled out in all that follows.]

[Section IV]

(Case ii)

$$\frac{\partial A_1}{\partial r_2}=\frac{(\rho_{23}\sigma_2-\sigma_3)\sigma_3}{k\sigma_2^2\sigma_3^2(1-\rho_{23}^2)}<0,\ \frac{\partial A_1}{\partial r_3}=\frac{(\rho_{23}\sigma_3-\sigma_2)\sigma_2}{k\sigma_{22}^2\sigma_3^2(1-\rho_{23}^2)}\gtreqless 0\Leftrightarrow \rho_{23}\sigma_3\gtreqless\sigma_2$$

$$\frac{\partial A_2}{\partial r_2}=\frac{1}{k\sigma_2^2(1-\rho_{23}^2)}>0,\ \frac{\partial A_2}{\partial r_3}=\frac{-\rho_{23}}{k\sigma_2\sigma_3(1-\rho_{23}^2)}\gtreqless 0\Leftrightarrow\rho_{23}\lesseqgtr 0$$

$$\frac{\partial A_3}{\partial r_2}=\frac{-\rho_{23}}{k\sigma_2\sigma_3(1-\rho_{23}^2)}\gtreqless 0\Leftrightarrow\rho_{23}\lesseqgtr 0,\ \frac{\partial A_3}{\partial r_3}=\frac{1}{k\sigma_3^2(1-\rho_{23}^2)}>0$$

$$\frac{\partial A_1}{\partial\sigma_2}=\frac{1}{k\sigma_2^3\sigma_3(1-\rho_{23}^2)}\{r_2(\sigma_3-\rho_{23}\sigma_2)+(r_2\sigma_3-r_3\rho_{23}\sigma_2)\}>0$$

(since $A_2>0$ requires that $(r_2\sigma_3-r_3\rho_{23}\sigma_2)>0$).

$$\frac{\partial A_1}{\partial\sigma_3}=\frac{1}{k\sigma_2\sigma_3^3(1-\rho_{23}^2)}\{r_3(\sigma_2-\rho_{23}\sigma_3)+(r_3\sigma_2-r_2\rho_{23}\sigma_3)\}$$

$$\frac{\partial A_2}{\partial\sigma_2}=\frac{1}{k\sigma_2^3\sigma_3(1-\rho_{23}^2)}r_3\rho_{23}\gtreqless 0\Leftrightarrow\rho_{23}\gtreqless 0$$

$$\frac{\partial A_2}{\partial\sigma_3}=\frac{1}{k\sigma_2^3\rho(1-\rho_{23}^2)}r_3\rho_{23}\gtreqless 0\Leftrightarrow\rho_{23}\gtreqless 0$$

$$\frac{\partial A_3}{\partial\sigma_2}=\frac{1}{(1-\rho_{23}^2)\sigma_2^2\sigma_3 k}r_2\rho_{23}\gtreqless 0\Leftrightarrow\rho_{23}\gtreqless 0$$

$$\frac{\partial A_3}{\partial\sigma_3}=\frac{-1}{k\sigma_2\sigma_3^3(1-\rho_{23}^2)}\{r_3\sigma_2+(r_3\sigma_2-r_2\rho_{23}\sigma_3)\}<0$$

These results may not be readily apparent, but if we assume $\rho_{23}\geqq 0$, all the expected signs follow, expept those for $\partial A_1/\partial r_3$ and $\partial A_1/\partial\sigma_3$ · Usually, however, all interest rates move in the same direction; so the assumption that $\rho_{23}\geqq 0$ is not unduly restrictive. To derive the expected signs for $\partial A_1/\partial r_3$ and $\partial A_1/\partial\sigma_3$, we must assume $\sigma_2\geqq\rho_{23}\sigma_3$. However, if we take into account the limited substitutabilility between long-term assets and money, these values are likely to be small in any event.

(Case i)

$$\frac{\partial A_2}{\partial r_2}=\frac{1}{k\alpha}>0,\ \frac{\partial A_2}{\partial r_3}=\frac{-1}{k\alpha}<0$$

$$\frac{\partial A_3}{\partial r_2}=\frac{-1}{k\alpha}<0,\ \frac{\partial A_3}{\partial r_3}=\frac{1}{k\alpha}>0$$

$$\frac{\partial A_2}{\partial\sigma_2}=\frac{1}{k\alpha^2}[k\{\sigma_3(\rho_{23}\sigma_2-\sigma_3)+\sigma_3(\rho_{23}\sigma_3-\sigma_2)\}-2(\sigma_2-\rho_{23}\sigma_3)(r_2-r_3)]\cdot$$

$$\frac{\partial A_2}{\partial\sigma_3}=\frac{-1}{k\alpha^2}[k\sigma_2\{\sigma_2(\rho_{23}\sigma_2-\sigma_3)+\sigma_3(\rho_{23}\sigma_3-\sigma_2)\}+2(\sigma_3-\rho_{23}\sigma_2)(r_2-r_3)]$$

$$\frac{\partial A_3}{\partial\sigma_2}=\frac{1}{k\alpha^2}\{k\sigma_3\{\sigma_2(\sigma_3-\rho_{23}\sigma_2)+\sigma_3(\sigma_2-\rho_{23}\sigma_3)\}+2(\sigma_2-\rho_{23}\sigma_3)(r_2-r_3)]$$

$$\frac{\partial A_3}{\partial \sigma_3}=\frac{-1}{k\alpha^2}[k\sigma_2\{\sigma_3(\rho_{23}\sigma_3-\sigma_2)+\sigma_2(\rho_{23}\sigma_2-\sigma_3)\}+2(r_2-r_3)(\sigma_3-\rho_{23}\sigma_2)]$$

In this case, signs with respect to the interest rates are definite and the expected ones, but those with respect to variance are by no means evident. This ambiguity remains even if we assume $\rho_{23}\geqq 0$, and $\sigma_2>\rho_{23}\sigma_3$. Only if we set $\sigma_2=\rho_{23}\sigma_3$, do we get the expected signs. However, when σ_2 tends to zero, the results become clear-cut:

$$\frac{\partial A_2}{\partial r_2}=\frac{1}{k\sigma_3^2}>0,\ \frac{\partial A_2}{\partial r_3}=\frac{-1}{k\sigma_3^2}<0$$

$$\frac{\partial A_2}{\partial \sigma_3}=\frac{2\sigma_3(r_3-r_2)}{k\sigma_3^2}>0,\ \frac{\partial A_3}{\partial \sigma_3}=\frac{2\sigma_3(r_2-r_3)}{k\alpha_3^2}<0$$

[Section V]

(Case ii)

$$\frac{\partial A_2}{\partial r_2}=\frac{(1+b)\sigma_3^2}{2b\Delta^2}\{(r_2\sigma_3-r_3\rho_{23}\sigma_2)^2-(1-\rho_{23}^2)\sigma_2^2(r_3^2+\sigma_3^2)\}$$

$$\frac{\partial A_2}{\partial r_3}=\frac{(1+b)\sigma_2\sigma_3}{2b\Delta^2}[\sigma_2\sigma_3\{2r_2r_3+\rho_{23}(1-\rho_{23}^2)\sigma_2\sigma_3\}-\rho_{23}(r_2^2\sigma_3^2+r_3^2\sigma_2^2)]$$

$$\frac{\partial A_3}{\partial r_2}=\frac{(1+b)\sigma_2\sigma_3}{2b\Delta^2}[\sigma_2\sigma_3\{2r_2r_3+\rho_{23}(1-\rho_{23}^2)\sigma_2\sigma_3\}-\rho_{23}(r_2^2\sigma_3^2+r_3^2\sigma_2^2)]$$

$$\frac{\partial A_3}{\partial r_3}=\frac{(1+b)\sigma_2^2}{2b\Delta^2}[(r_3\sigma_2-r_2\rho_{23}\sigma_3)^2-(1-\rho_{23}^2)\sigma_3^2(r_2^2+\sigma_2^2)]$$

$$\frac{\partial A_1}{\partial r_2}=-\left(\frac{\partial A_2}{\partial r_2}+\frac{\partial A_3}{\partial r_2}\right),\ \frac{\partial A_1}{\partial r_3}=-\left(\frac{\partial A_2}{\partial r_3}+\frac{\partial A_3}{\partial r_3}\right)$$

$$\frac{\partial A_2}{\partial \sigma_2}=\frac{(1+b)\sigma_3}{2b\Delta^2}[2r_2\sigma_2\sigma_3\{r_3^2+\sigma_3^2(1-\rho_{23}^2)\}-r_3\rho_{23}\{r_2^2\sigma_3^2+r_3^2\sigma_2^2+\sigma_2^2\sigma_3^2(1-\rho_{23}^2)\}]$$

$$\frac{\partial A_2}{\partial \sigma_3}=\frac{-(1+b)}{b\Delta}\{r_3\rho_{23}\sigma_2^2\sigma_3^2(1-\rho_{23}^2)\}$$

$$\frac{\partial A_3}{\partial \sigma_2}=\frac{-(1+b)}{b\Delta}\{r_2\rho_{23}\sigma_2^2\sigma_{23}^2(1-\rho_{23}^2)\}$$

$$\frac{\partial A_3}{\partial \sigma_3}=\frac{(1+b)\sigma_2}{2b\Delta}[2r_3\sigma_2\sigma_3\{r_2^2+\sigma_2^2(1-\rho_{23}^2)\}-r_2\rho_{23}\{r_2^2\sigma_3^2+r_3^2\sigma_2^2+\sigma_2^2\sigma_3^2(1-\rho_{23}^2)\}]$$

$$\frac{\partial A_1}{\partial \sigma_2}=-\left(\frac{\partial A_2}{\partial \sigma_2}+\frac{\partial A_3}{\partial \sigma_2}\right),\ \frac{\partial A_1}{\partial \sigma_3}=-\left(\frac{\partial A_2}{\partial \sigma_3}+\frac{\partial A_3}{\partial \sigma_3}\right)$$

The signs of none of these derivatives is evident, a fact that suggests another reason to avoid the use of the quadratic utility function.

(Case i)

$$\frac{\partial A_2}{\partial r_2}=\frac{-1}{2\delta}[(1+b)\{\alpha-(r_2-r_3)^2\}+2b\{r_3(\sigma_2^2-\sigma_3^2)-r_3(r_2-r_3)^2$$
$$+2r_2\sigma_3(\sigma_3-\rho_{23}\sigma_2\}]$$

$$\frac{\partial A_2}{\partial r_3}=\frac{1}{2\delta}[(1+b)\{\alpha-(r_2-r_3)^2\}+2b\{r_2(\sigma_3^2-\sigma_2^2)-r_2(r_2-r_3)^2$$
$$+2r_3\sigma_2(\sigma_2-\rho_{23}\sigma_3)\}]$$

$$\frac{\partial A_3}{\partial r_2}=\frac{-1}{2\delta}[(1+b)\{\alpha-(r_2-r_3)^2+2b\{r_3(\sigma_2^2-\sigma_3^2)-r_3(r_2-r_3)^2 2r_2\sigma_3(\sigma_3$$
$$-\rho_{23}\sigma_2)\}]$$

$$\frac{\partial A_3}{\partial r_3}=\frac{1}{2\delta}[(1+b)\{\alpha-(r_2-r_3)^2\}+2b\{r_2(\sigma_3^2-\sigma_2^2)-r_2(r_2-r_3)^2+2r_3\sigma_2(\sigma_2$$
$$-\rho_{23}\sigma_3)\}]$$

where $\delta\equiv b\{\alpha+(r_2-r_3)^2\}^2$

$$\frac{\partial A_2}{\partial \sigma_2}=\frac{-1}{\delta}[b\{\sigma_2\sigma_3(\sigma_3-\rho_{23}\sigma_2)+\sigma_3^2(\sigma_2-\rho_{23}\sigma_3)+\rho_{23}\sigma_3(r_2^2-r_3^2)$$
$$-2r_3(\sigma_2-\sigma_3\rho_{23})(r_2-r_3)\}-(1+b)(r_2-r_3)(\sigma_2-\rho_{23}\sigma_3)]$$

$$\frac{\partial A_2}{\partial \sigma_3}=\frac{-1}{\delta}[b\{\sigma_2^2(\rho_{23}\sigma_2-\sigma_3)+\sigma_2\sigma_3(\rho_{23}\sigma_3^2-\sigma_2)+(r_2-r_3)$$
$$\times(r_2\rho_{23}\sigma_2+r_3\rho_{23}\sigma_2-2r_2\sigma_3)-(1+b)(r_2-r_3)(\sigma_3-\rho_{23}\sigma_2)]$$

$$\frac{\partial A_3}{\partial \sigma_2}=\frac{-1}{\delta}[(1+b)(r_2-r_3)(\sigma_2-\rho_{23}\sigma_3)-b\{\sigma_2\sigma_3+(\sigma_3-\rho_{23}\sigma_2)+\sigma_3^2(\sigma_2-\rho_{23}\sigma_3)$$
$$+(r_2-r_3)(r_2\rho_{23}\sigma_3+r_3\rho_{23}\sigma_3-2r_3\sigma_2)]$$

$$\frac{\partial A_3}{\partial \sigma_3}=\frac{-1}{\delta}[b\{\sigma_2^2(\rho_{23}\sigma_2-\sigma_3)+\sigma_2\sigma_3(\rho_{23}\sigma_3-\sigma_2)+(r_2-r_3)$$
$$\times(r_2\rho_{23}\sigma_2+r_3\rho_{23}\sigma_2-2r_2\sigma_3)\}-(1+b)(r_2-r_3)(\sigma_3-\rho_{23}\sigma_2)]$$

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